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Spectral Decomposition of Uniformly Elliptic Multiparameter Eigenvalue Problems

P. A. BINDING*

Department of Mathematics, University of Calgary, Calgary, Canada T2N 1N4

AND

B. D. SLEEMAN

Department of Mathematics, University of Dundee, Dundee DD1 4HN, Scotland

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1. INTRODUCTION

Expansion results for self-adjoint multiparameter eigenvalue problems of the form

$$W_m(\boldsymbol{\lambda})\, x_m = 0 \neq x_m, \tag{1.1}$$

where $W_m(\boldsymbol{\lambda}) = T_m - \sum_{n=1}^{k} \lambda_n V_{mn}$ act on separable Hilbert spaces, H_m date back to the work of Dixon and Hilbert early this century. The original motivation was from the separation of variables technique, applied to linear partial differential equations (pde), leading to Sturm–Liouville equations in which

$$T_m y = -(p_m y')' + q_m y$$

for a suitable domain, with p_m positively bounded below. One seeks to show that spectral expansions via the original pde can also be derived via the separated ode, thus "justifying" the separation technique in one sense. In this paper we adopt a more modern abstract setting which includes the Sturm–Liouville case and various generalisations, as well as finite dimen-

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sional approximations. Elimination of all λ_i except λ_n from (1.1) leads formally to operator equations of the form

$$\Delta_n x = \lambda_n \Delta_0 x, \tag{1.2}$$

where $x = x_1 \otimes \cdots \otimes x_k$ and the operators Δ_j $(0 \leqslant j \leqslant k)$, which are defined more carefully below, are operators in the Hilbert space tensor product $H = \bigotimes_{m=1}^{k} H_m$. It is convenient to classify (1.1) in terms of consequent properties of (1.2).

Formally we may write

$$\Delta_n = T_1 \otimes C_{1n} + \cdots + C_{kn} \otimes T_k,$$

where the C_{ij} are the cofactors of $\Delta_0 = \otimes \det[V_{mn}]$. We refer to (1.2) as right definite (RD) if $\Delta_0 > 0$ and as uniformly right definite (URD) if $\Delta_0 \gg 0$. Here $A > 0$ $[\gg 0]$ means $(x, Ax) > \alpha(x, x)$ for $\alpha = 0$ $[>0]$ and all nonzero $x \in D(A)$.

Most applications to separation of variables are RD but not URD, and this complicates the analysis considerably. Many applications are, however, uniformly elliptic (UE), which means each cofactor $C_{ij} \gg 0$. If (as we assume throughout) each T_n is bounded below, then under UE each Δ_n is also bounded below and is a uniformly elliptic partial differential operator in the Sturm–Liouville case of (1.1). We remark that although these "definiteness conditions" are expressed in terms of (1.2), they are equivalent to analogous conditions expressed in terms of (1.1)—see [2].

The above definiteness conditions on the V_{mn} are usually supplemented by one or both of the following conditions on the T_m:

(i) compactness (of the resolvents of the T_m).

This frequently holds when the underlying problem (e.g. a Schrödinger equation) is posed in a bounded domain, but see [10] for an application to inverse scattering theory. Generally, unbounded domains (e.g., "exterior" problems) correspond to failure of compactness.

(ii) positivity (of the T_m, i.e., $T_m > 0$).

This was an essential ingredient of UE expansion theory until recently, but it is very problem dependent, e.g., on boundary conditions. Early results used both compactness and positivity. Then each $T_m \gg 0$, so each $\Delta_n \gg 0$, and the problem is then called uniformly left definite (ULD)—cf. (1.2).

Assuming compactness, several authors (e.g., Cordes, Atkinson, Faierman, Browne, Volkmer) have given right definite expansion theorems under various additional hypotheses, and we refer to Volkmer [20] for several new results, and for a fine review of the whole area. Although the works of Dixon and Hilbert, referred to earlier, were for the two parameter

ULD case, it was not until 1976 that the k-parameter analogue was treated by Källström and Sleeman [14]. This was generalized to the operator context here (but still under ULD) by Binding [3], with expansion in metrics equivalent to $W^{2,2}$ for the Sturm–Liouville case. Again see Volkmer [20] for various left definite results even without uniformity. Faierman (e.g., [12]) has given both URD and ULD expansion theorems (mostly for $k=2$ Sturm–Liouville equations), but in the metric of uniform convergence. These require additional smoothness conditions of the coefficients and of the functions to be expanded, and recent improvements can be found (for general k) in the work of Rynne [16].

The UE case has been treated without definiteness (but still with compactness) only recently. In this case H must be decomposed into $F+G$, where $\dim F$ is finite and eigenfunction expansions are valid in G. See Faierman [13] for $k=2$ Sturm–Liouville equations and uniform convergence, and Binding and Seddighi [6] for k equations and (abstract) $W^{2,2}$ convergence. Further information on the "Jordan" structure of F can be found in [5, 13].

Without compactness, series expansions must be replaced by integrals, and it is usual to state the results in terms of decomposition by spectral measures. In 1955, Cordes [11] gave such a decomposition for the Stark effect, but this analytical tour de force made use of rather special assumptions, including RD and ULD for $k=2$ Sturm–Liouville equations. Incidentally, Cordes's theory has been reworked in modern terms by McGhee and Picard [15], and this provides a nice basis for further work. The general URD case has been treated by Browne [9] and Volkmer [19], but the RD case is still open. Spectral decomposition has been discussed by Sleeman [18] and Binding [4] under a definiteness condition including URD and ULD.

Here we assume UE but we relax both the positivity and compactness conditions. The Δ_n remain uniformly elliptic in the Sturm–Liouville case, but in general with at least partly continuous spectrum. Our results unify a variety of completeness theorems for separable boundary value problems, in particular when $\sigma(\Delta_n)$ is either discrete or positive for some n. To illustrate our results, we close with an example where part of the support of the spectral measure consists of two eigencurves. This has some of the features of Cordes's work [11] but with much less technicality. We also show how to rewrite the spectral expansions via integrals along these curves, and in the future we hope to extend this aspect of the work to problems with infinitely many eigencurves.

Let us briefly motivate and describe our analysis. In the ULD case with compact T_m^{-1}, the operators $B_n=\Delta_n^{-1}\Delta_0$ are compact and commuting and this forms the basis of the analysis of [3]. If the T_m are bounded below with compact resolvents, then the operators $B_n(\xi, \varepsilon)$ defined as above but with T_m replaced by $T_m + \xi I_m + \varepsilon \sum_{n=1}^{k} V_{mn}$ enjoy similar properties for certain $(\xi, \varepsilon) \in \mathbb{C} \times \mathbb{R}$. This, together with the holomorphy of $B_n(\xi, \varepsilon)$ in ξ, forms the basis of the expansion in [7]. The analysis depends heavily on discrete spectra at various points, however, and we have focussed our attention instead on the operators (formally) defined by $\Gamma_n = \Delta_1^{-1} \Delta_n$, which are self-adjoint and commuting in an appropriate Hilbert space D_0 provided $\Delta_1 \gg 0$ [4]. In our case Δ_1 may be indefinite and in Section 2 we use parametric variation techniques to define $\Gamma_n(\xi, \varepsilon)$ analogously to $B_n(\xi, \varepsilon)$ in [7]. Holomorphy in ξ and a spectral decomposition of what is now a Pontryagin space D_0 are discussed in Section 3. The primary difficulty is that, for $n > 1$, $\Gamma_n(\xi, \varepsilon)$ involves Δ_1 and Δ_n, both of which are parametrically varying and unbounded.

2. The Operators $\Gamma_n(\xi)$

We begin with a more precise statement of our basic assumptions. We are given self-adjoint operators T_m, V_{mn}, where the V_{mn} are bounded and T_m is bounded below. With $^+$ denoting induced self-adjoint operators in H, we define

$$\Delta_0 = \det[V_{mn}^+],$$

and Δ_{0mn} as the corresponding (m, n) cofactor. These operators are self-adjoint on H and we assume that Δ_0 is 1–1 and that each $\Delta_{0mn} \gg 0$. For a discussion of these assumptions see [7, Section 1]. In Section 3 we make an additional technical assumption.

Now define $D = \bigcap_{m=1}^{k} D(T_m^+)$. Then the operator

$$\Delta_n|_D = \sum_{m=1}^{k} \Delta_{0mn} T_m^+$$

has a self-adjoint closure, which we denote by Δ_n [19, Lemma 4.4]. We assume that for some $\boldsymbol{\omega} \in \mathbb{R}^k$, $\inf \sigma_e \left(\sum_{n=1}^{k} \omega_n \Delta_n\right) > 0$. By a rotation of the $\boldsymbol{\lambda}$-axes we may and shall assume $\boldsymbol{\omega} = (1, 0, \ldots)$ and so

$$\Delta_1 = \Pi_1 - \Phi_1, \tag{2.1}$$

where $\Pi_1 \gg 0$ and Φ_1 has finite rank.

Lemma 2.1. *For some $\varepsilon_0 > 0$, $\Delta_1 + \varepsilon \Delta_0$ is boundedly invertible whenever $0 < |\varepsilon| < \varepsilon_0$.*

Proof. If Δ_1 is boundedly invertible, the result is obvious. Suppose

$\Delta_1 u_j \to 0$ for some unit $u_j \in H$ and let $u_j \rightharpoonup x$. Then $u_j - \Pi_1^{-1}\Phi_1 u_j \to 0$ and since $\Pi_1^{-1}\Phi_1$ is compact both summands converge strongly. In particular

$$x - \Pi_1^{-1}\Phi_1 x = 0,$$

whence $\Delta_1 x = 0$. Now $N(\Delta_1) = N(I - \Pi_1^{-1/2}\Phi_1 \Pi_1^{-1/2})$ has finite dimension and so we may proceed as with the proof of [7, Theorem 2.4] to establish the result.

Translating λ_1 to $\lambda_1 + \varepsilon$, if necessary, we can therefore ensure that the analogue of Δ_1, viz. $\Delta_1 + \varepsilon\Delta_0$, has a bounded inverse. To simplify notation we assume this translation to have been made at the outset and consequently we assume, without loss of generality, that Δ_1 has a bounded inverse.

We now define D_0 as $D(\Pi_1^{1/2})$ under the inner product given by

$$[x, y]_0 = (\Pi_1^{1/2} x, \Pi_1^{1/2} y). \tag{2.2}$$

As in [7] we now replace T_m by $T_m + \xi I_m$, where $\xi \in \mathbb{C}$, leading to an analogue

$$\Delta_n(\xi) = \Delta_n + \xi \Sigma_n, \qquad n = 1, ..., k$$

of Δ_n, where $\Sigma_n = \sum_{m=1}^{k} \Delta_{0mn} \gg 0$.

Remark 2.2. D_0 is the completion of D in the inner product given by any of the (topologically equivalent) inner products generated by operators of the form $\Pi_1 + E$, where E is bounded and $\Pi_1 + E \gg 0$.

We next show that

$$\Gamma_n(\xi)|_D = \Delta_1(\xi)^{-1} \Delta_n(\xi)|_D$$

is D_0 bounded and hence has an extension by continuity to D_0, for at least two sets of $\xi \in \mathbb{C}$. We write $\xi = \rho + i\sigma$, $(\rho, \sigma) \in \mathbb{R}^2$. There are two cases to consider, namely $\xi \in \mathbb{R}$, $\xi \in \mathbb{C}\backslash\mathbb{R}$.

Case 1. Let $\sigma = 0$ and choose ρ so that

(i) $M_\rho := \Pi_1 + \rho\Sigma_1 \gg 0$ and

(ii) $\Delta_1(\rho)$ has a bounded inverse on H.

The set of ρ such that (ii) holds contains sufficiently small intervals centred either at 0 or at ρ_0 chosen positive enough to ensure that $\Delta_1(\rho_0) \gg 0$. Define D_ρ as the completion of D under the inner product $[\cdot, \cdot]_\rho$ generated by M_ρ. By Remark 2.2, D_ρ is homeomorphic to D_0. Since all $\Delta_{0mn} \geqslant 0$ and are bounded and since

$$[M_\rho^{-1}\Delta_n(\rho)\,x, x]_\rho = (\Delta_n(\rho)\,x, x) \qquad \forall x \in D$$

there exist $\beta > 0$ and $\gamma \in \mathbb{R}$ such that

$$\alpha[x, x]_\rho \leqslant [M_\rho^{-1}\Delta_n(\rho)\,x, x]_\rho \leqslant \beta[x, x]_\rho \tag{2.3}$$

$\forall x \in D$, $n = 1, ..., k$. By polarisation we see that $M_\rho^{-1}\Delta_n(\rho)$ generates a D_ρ-bounded sesquilinear form and hence a D_0-bounded operator on D.

Moreover $M_\rho^{-1}\Delta_1(\rho) = I - M_\rho^{-1}\Phi_1$ is $1-1$. Since $M_\rho^{-1}\Phi_1$ is compact on D_ρ, $M_\rho^{-1}\Delta_1(\rho)$ is therefore D_0-boundedly invertible and so

$$\Gamma_n(\rho)|_D = (M_\rho^{-1}\Delta_1(\rho))^{-1}\, M_\rho^{-1}\Delta_n(\rho)|_D$$

is D_0 bounded.

Case 2. Now suppose $\sigma \neq 0$ and choose ρ so that $M_\rho \gg 0$. Thus for some $\varepsilon > 0$, all ξ such that $\rho \geqslant -\varepsilon$ and $\sigma \neq 0$ are allowed. Unless otherwise stated we work entirely with the D_ρ inner product. As in the first case we see that $M_\rho^{-1}\Delta_n(\xi)$ is bounded on D. We now claim that

$$M_\rho^{-1}\Delta_1(\xi) = I - M_\rho^{-1}\Phi_1 + i\sigma M_\rho^{-1}\Sigma_1$$

has a bounded inverse. Indeed suppose

$$M_\rho^{-1}\Delta_1(\xi)\,x_j \to 0$$

and, without loss of generality, that $x_j \rightharpoonup x$. Since Φ_1 is compact

$$\begin{aligned} i\sigma(M_\rho^{-1}\Sigma_1 - i\sigma^{-1}I)\,x_j &= (I + i\sigma M_\rho^{-1}\Sigma_1)\,x_j \\ &\to M_\rho^{-1}\Phi_1 x. \end{aligned}$$

Moreover $M_\rho^{-1}\Sigma_1$ is self-adjoint, so $M_\rho^{-1}\Sigma_1 - i\sigma^{-1}I$ is boundedly invertible and therefore

$$x_j \to (I + i\sigma M_\rho^{-1}\Sigma_1)^{-1}\, M_\rho^{-1}\Phi_1 x.$$

Thus the right-hand side equals x and on rearrangement we obtain

$$y := (I - M_\rho^{-1}\Phi_1)\,x + i\sigma M_\rho^{-1}\Sigma_1 x = 0.$$

Thus $0 = \operatorname{Im}[x, y]_\rho = \sigma(x, \Sigma_1 x)$, where $(\cdot, \cdot)$ is the H inner product. Since $\Sigma_1 \gg 0$ we have $x = 0$ and our claim is established.

Finally then

$$\Gamma_n(\xi)|_D = (M_\rho^{-1}\Delta_1(\xi))^{-1} M_\rho^{-1}\Delta_n(\xi)$$

is D_ρ and hence D_0-bounded.

3. Spectral Decomposition

The main tool for our investigation is the set of operators

$$\Gamma_n = \Gamma_n(0), \qquad n = 0, 1, ..., k,$$

where $\Gamma_n(\xi)$ is the extension by continuity of $\Gamma_n(\xi)|_D$ to D_0—see Remark 2.2.

THEOREM 3.1. *The Γ_n are bounded self-adjoint and pairwise commutative on D_0.*

Proof. Only the final contention requires proof. Choose ρ_0 so that $\Delta_1(2\rho_0) \gg 0$ and, with $\Omega(c, \varepsilon)$ denoting a disk in $\mathbb{C}$ with centre c and radius $\varepsilon > 0$, write Z_ε for the interior of

$$\Omega(0, \varepsilon) \cup \Omega(2\rho_0, \varepsilon) \cup (\Omega(\rho_0, \rho_0 + \varepsilon) \backslash \Omega(\rho_0, \rho_0 - \varepsilon)), \tag{3.1}$$

cf. [7, Eq. (3.1)]. For small enough ε, the results of Section 2 show that the $\Gamma_n(\xi)$ are bounded operators on D_0 for all $\xi \in Z_\varepsilon$. We claim that $\Gamma_n(\xi)$ is holomorphic in Z_ε. Indeed $(u, \Gamma_n(\xi) v)_0 = (u, (I - \Phi_1 \Pi_1^{-1} + \xi \Sigma_1 \Pi_1^{-1})^{-1} \Delta_n(\xi) v)$ which can be expanded in a power series since

$$I - \Phi_1 \Pi_1^{-1} \text{ is boundedly invertible.} \tag{3.2}$$

This follows since Φ_1 is compact and $(I - \Phi_1 \Pi_1^{-1}) x = 0$ implies that $\Delta_1 y = 0$, where $y = \Pi_1^{-1} x$ and so $y = 0$ which implies that $x = 0$.

The argument of [7, Theorem 3.1] can now be applied to the equations

$$\Gamma_n(\xi)\, \Gamma_m(\xi) = \Gamma_m(\xi)\, \Gamma_n(\xi), \qquad \xi \in Z_\varepsilon.$$

The general spectral decomposition is now within reach. From (2.1) D_0 is a Pontryagin space under the indefinite inner product given by

$$(x, y)_0 = [x, y]_0 - (x, \Phi_1 y)$$

(see 2.2). Note that $(x, y)_0 = (x, \Delta_1 y)$ when $y \in D(\Delta_1)$.

We now introduce the following technical assumption.

Assumption 3.2. Any root subspace S of Γ_0 corresponding to a real eigenvalue is nondegenerate in D_0.

This means that the Gram operator corresponding to $(\,,\,)_0$ is invertible on S: see [6, p. 119; 8, p. 9]. The assumption holds trivially if $\sigma(\Gamma_0) \cap \mathbb{R}$ is entirely continuous, i.e., if Γ_0 has no real eigenvalues. It also holds if $\sigma(\Delta_1)$ is positive or discrete (cf. [6]), thus covering most of the cases in the literature, and for certain differential operators Δ_1 with the unique continuation property; we hope to report on this elsewhere.

THEOREM 3.3. *Under Assumption* 3.2, D_0 *admits a* $(\cdot, \cdot)_0$*-orthogonal decomposition* $F \oplus G$, *where* F, G *are invariant for each* Γ_n, $\dim F \leqslant 3 \operatorname{rank} \Phi_1$ *and* G *is a Hilbert space under* $(\cdot, \cdot)_0$. D_0 *therefore admits*

(i) *a basis of joint root vectors for the* $\Gamma_n|_F$.

(ii) *a joint resolution* $\int_\sigma E(d\boldsymbol{\mu})$ *of* $\Gamma_1|_G = I|_G$ *so that*

$$f(\Gamma_0, \Gamma_2, \ldots, \Gamma_k) = \int_\sigma f(\boldsymbol{\mu})\, E(d\boldsymbol{\mu})$$

for Borel functions f *on* σ, *where* $\boldsymbol{\mu} \in \sigma$ *is equivalent to* $(\Gamma_n - \mu_n)\, x_j \to 0$ *in* D_0 *for some* (*n-independent*) $x_j \not\to 0$ *in* D_0.

Proof. The arguments of [6, Corollaries 4, 5] give a Γ_0 invariant decomposition $F \oplus G$. We remark that the cited conclusion is stronger and depends on compactness of Δ_1^{-1}, but Assumption 3.2 suffices for our weaker conclusion.

Now F is a direct sum of subspaces of the form $S = N(\Gamma_0 - \lambda I)^l$ where $l \leqslant 1 + 2 \operatorname{rank} \Phi_1$ [8, p. 191]. By Theorem 3.1, Γ_n and $(\Gamma_0 - \lambda I)^l$ commute, so S is also Γ_n-invariant and we may decompose it into joint root subspaces for the Γ_n; cf. [7, Lemma 4.2]. Moreover $G = F^\perp$ is invariant for Γ_n by self adjointness and so the final contention comes from standard spectral theory applied to the $\Gamma_n|_G$.

It remains to relate the above constructions to the original problem (1.1). Expressions for a basis of F in terms of the T_m and V_{mn} have been given elsewhere [5], so here we concentrate on analogues of [4, Sections 7, 8]. Let

$$\phi(\mu_0, 1, \mu_2, \ldots, \mu_k) = \mu_0^{-1}(1, \mu_2, \ldots, \mu_k) \in \mathbb{C}^k.$$

THEOREM 3.4. *If* $0 \in \bigcap_{m=1}^k \sigma(W_m(\boldsymbol{\lambda}))$ *then* $\lambda_1^{-1} \in \sigma(\Gamma_0)$ *and* $\lambda_1^{-1}\lambda_n \in \sigma(\Gamma_n)$, $n = 1, \ldots, k$. *Conversely if* $\boldsymbol{\mu} \in \sigma$ (*of Theorem* 3.3) *then* $0 \in \bigcap_{m=1}^k \sigma(W_m(\phi(\boldsymbol{\mu})))$. *Moreover spectrum may be replaced by point spectrum in these results, and any joint eigenspaces for the* Γ_n *in* G (*see Theorem* 3.3) *are spanned by decomposable elements* $x_1 \otimes \cdots \otimes x_k$, *where* x_m *satisfy* (1.1).

Proof. If $0 \in \bigcap_{m=1}^{k} \sigma(W_m(\boldsymbol{\lambda}))$ then $W_m(\boldsymbol{\lambda})\, u_m^j \to 0$ in H_m for some unit $u_m^j \in H_m$. Thus $W_m(\boldsymbol{\lambda})^+\, u^j \to 0$ in H for unit $u^j = u_1^j \otimes \cdots \otimes u_k^j$, $^+$ denoting induced self-adjoint operator. It follows readily that

$$(\Delta_n - \lambda_n \Delta_0)\, u^j \to 0, \qquad u = 1, ..., k. \tag{3.3}$$

We claim that Δ_1^{-1} is a bounded operator from H to D_0. From (2.1), (2.2), and (3.2),

$$\Delta_1^{-1} = \Pi_1^{-1}(I - \Phi_1 \Pi_1^{-1})^{-1},$$

so

$$|[x, \Delta_1^{-1}x]_0| = |(x, (I - \Phi_1 \Pi_1^{-1})\, x)| \leqslant \beta[x, x]_0,$$

for some β, since Π_1^{-1} is bounded. The claim thus follows as for (2.3) et seq., and (3.3) for $n = 1$ yields

$$(I - \lambda_1 \Gamma_0)\, u^j \to 0 \qquad \text{in } D_0.$$

Since $u^j \rightharpoonup 0$ in H, $u^j \not\to 0$ in D_0, so $\lambda_1 \neq 0$ and $\lambda_1^{-1} \in \sigma(\Gamma_0)$. From (3.3) again

$$(\Delta_n - \lambda_1^{-1}\lambda_n \Delta_1)\, u^j \to 0$$

whence

$$(\Gamma_n - \lambda_1^{-1}\lambda_n I)\, u^j \to 0 \qquad \text{in } D_0$$

and we have $\lambda_1^{-1}\lambda_n \in \sigma(\Gamma_n)$.

Conversely we have

$$\begin{aligned}(I + (\xi - \alpha)(T_m^+ + \alpha I)^{-1})\, \Gamma_0(\xi) \\ = \sum_{n=1}^{k} (T_m^+ + \alpha I)^{-1}\, V_{mn}^+ \Gamma_n(\xi), \qquad m = 1, ..., k\end{aligned}$$

as the analogue of [7, Eq. (3.3)], valid for all $\xi \in Z$—see (3.1). Setting $\xi = 0$ we obtain

$$T_m^+ \Gamma_0 = \sum_{n=1}^{k} V_{mn}^+ \Gamma_n. \tag{3.4}$$

If $\boldsymbol{\mu} \in \sigma$, say $x_j \not\to 0$ in D_0 but $(\Gamma_n - \mu_n)\, x_j \to 0$ in D_0 then

$$(\Gamma_n - \mu_n)\, x_j \to 0, \tag{3.5}$$

in H. Thus

$$(W_m(\mu_0^{-1}\boldsymbol{\mu}))^+\, \Gamma_0 x_j = \sum_{n=1}^{k} V_{mn}^+(\Gamma_n - \mu_0^{-1}\mu_n \Gamma_0)\, x_j$$

By (3.4). Using (3.5) we reach

$$(W_m(\mu_0^{-1}\boldsymbol{\mu}))^+ \Gamma_0 x_j \to 0 \qquad \text{in } H. \tag{3.6}$$

Suppose $0 \notin \sigma((W_m(\mu_0^{-1}\boldsymbol{\mu}))^+) = \sigma(W_m(\mu_0^{-1}\boldsymbol{\mu}))$. Then $\Gamma_0 x_j \to 0$, so $T_m^+ \Gamma_0 x_j \to 0$ in H by (3.6), so $\Pi_1 \Gamma_0 x_j \to 0$ in H by (2.1) and finally

$$[\Gamma_0 x_j, \Gamma_0 x_j]_0 = (\Gamma_0 x_j, \Pi_1 \Gamma_0 x_j) \to 0.$$

It follows that $\Gamma_0 x_j \to 0$ in D_0, so $\lambda_0 x_j \to 0$ in D_0 which contradicts $x_j \to 0$ in D_0.

This completes the proof of the first two contentions and the others follow as for [4, Theorem 8.2 and Corollary 8.3].

4. Application

As an illustration of the main results of this paper we consider the following abstract two parameter problem

$$(T_1 - \lambda_1 V_{11} - \lambda_2 V_{12})\, x_1 = 0 \neq x_1 \in H_1 \tag{4.1}$$

$$(T_2 - \lambda_1 V_{21} - \lambda_2 V_{22})\, x_2 = 0 \neq x_2 \in H_2, \tag{4.2}$$

where

$$H_1 = L_2[0, \infty),\ D(T_1) = \{y \in H_1 : y'' \in H_1,\ y'(0) + \theta y(0) = 0\}, \qquad \theta < 0,$$

$T_1 y = \varepsilon y - y''$, $V_{11} = 3I_1/4$, $-V_{12} = I_1 =$ identity in H_1, and where $H_2 = \mathbb{C}^2$ with vector—matrix representation and

$$T_2 z = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} z, \qquad V_{21} z = \begin{bmatrix} -1 & 1/2 \\ 1/2 & -1 \end{bmatrix} z, \qquad V_{22} = I_2 = \text{identity in } H_2.$$

It is clear that T_m, V_{mn} are self adjoint with T_m bounded below, V_{mn} bounded, V_{mn}^{-1} bounded, and $(-1)^{m+n} V_{mn} > 0$, $1 \leqslant m, n \leqslant 2$. We also note that the point spectrum

$$\sigma_p(T_1) = \{\tau\} \qquad \text{where} \quad \tau = \varepsilon - \theta^2$$

and the continuous spectrum

$$\sigma_c(T_1) = \sigma_e(T_1) = [\varepsilon, \infty), \tag{4.3}$$

where σ_e denotes essential spectrum; see, e.g., [1, pp. 204–5]. Thus (4.1) has nontrivial resolvent for

$$3\lambda_1/4 - \lambda_2 \in \{\tau\} \cup [\varepsilon, \infty) \tag{4.4}$$

and for (4.2) we require

$$0=\det\begin{bmatrix}1-\lambda_1+\lambda_2 & \lambda_1/2\\ \lambda_1/2 & -1-\lambda_1+\lambda_2\end{bmatrix}=(\lambda_2-\lambda_1)^2-1-\lambda_1^2/4.$$

This can be rewritten

$$(\lambda_2-3\lambda_1/2)(\lambda_2-\lambda_1/2)=1. \tag{4.5}$$

The curves represented by (4.4), (4.5) are depicted in Fig. 1. The spectrum σ of (4.1), (4.2) is given by

$$\begin{aligned}\sigma &= \{(\lambda_1,\lambda_2)\in\mathbb{C}^2\colon (4.1), (4.2) \text{ are both satisfied}\}\\ &= \{P_\tau^-\}\cup\{P_\tau^+\}\cup \operatorname{arc} P_\varepsilon^- P_\infty^- \cup \operatorname{arc} P_\varepsilon^+ P_\infty^+.\end{aligned}$$

Note that at this stage $P_\varepsilon^\pm P_\tau^\pm$ need be neither real nor distinct.

We now construct the Hilbert space tensor product $H=H_1\otimes H_2$, where

$$(u\otimes v,\ y\otimes z)_H=(u,y)_{H_1}\,(v,z)_{H_2}$$

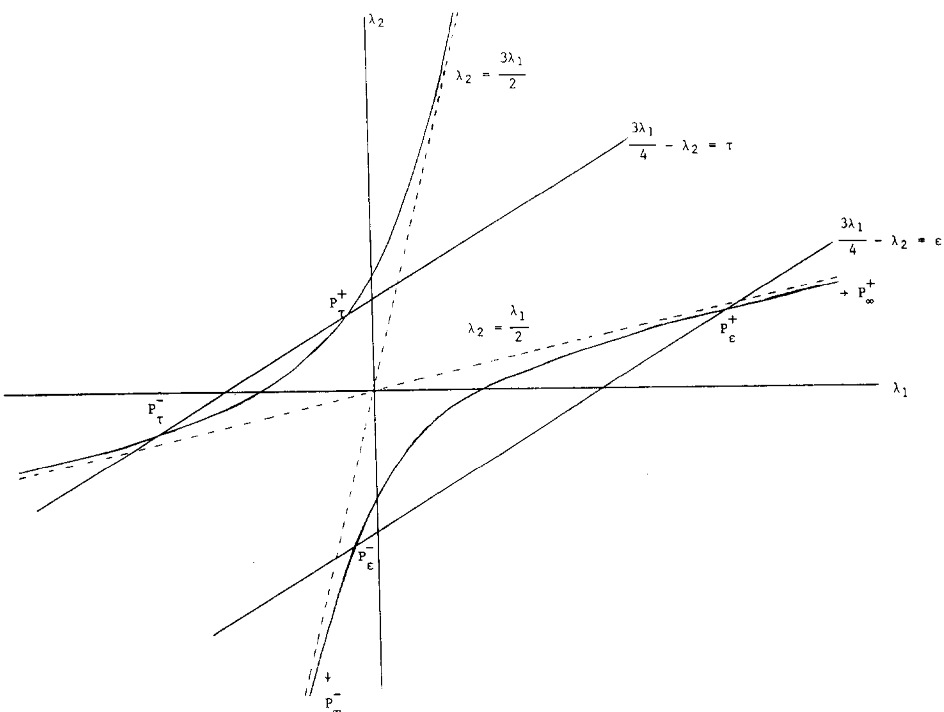

FIGURE 1

and define

$$\begin{aligned} \Delta_0 &= V_{11} \otimes V_{22} - V_{12} \otimes V_{21} && \text{on } H, \\ \Delta_1 &= T_1 \otimes V_{22} - V_{12} \otimes T_2 && \text{on } D(T_1) \otimes H_2, \\ \Delta_2 &= V_{11} \otimes T_2 - T_1 \otimes V_{21} && \text{on } D(T_1) \otimes H_2. \end{aligned}$$

On using the isomorphism $H \to H_1^2$: $y_1 \otimes [{}^1_0] + y_2 \otimes [{}^0_1] \to [{}^{y_1}_{y_2}]$ we may represent the Δ_n as

$$\Delta_0 = \begin{bmatrix} 3I_1/4 & 0 \\ 0 & 3I_1/4 \end{bmatrix} + \begin{bmatrix} -I_1 & I_1/2 \\ I_1/2 & -I_1 \end{bmatrix} = \begin{bmatrix} -I_1/4 & I_1/2 \\ I_1/2 & -I_1/4 \end{bmatrix}$$

$$\Delta_1 = \begin{bmatrix} T_1 & 0 \\ 0 & T_1 \end{bmatrix} + \begin{bmatrix} -I_1 & 0 \\ 0 & I_1 \end{bmatrix} = \begin{bmatrix} T_1 - I_1 & 0 \\ 0 & T_1 + I_1 \end{bmatrix}$$

$$\Delta_2 = \begin{bmatrix} T_1 & -T_1/2 \\ -T_1/2 & T_1 \end{bmatrix} + \begin{bmatrix} -3I_1/4 & 0 \\ 0 & 3I_1/4 \end{bmatrix} = \begin{bmatrix} T_1 - 3I_1/4 & -T_1/2 \\ -T_1/2 & T_1 + 3I_1/4 \end{bmatrix}.$$

We note that

$$\Delta_0 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

implies that

$$\begin{bmatrix} -1/4 & 1/2 \\ 1/2 & -1/4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and so Δ_0 is 1–1. Similarly Δ_0 is indefinite. If Δ_0 were definite then the problem could be treated by standard methods for the URD case—see Section 1. Computing $N(\Delta_n)$ as for $N(\Delta_0)$ we find

$$\Delta_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{is equivalent to} \quad T_1 x = x, \qquad T_1 y = -y$$

which implies that $\tau = \pm 1$. Similarly $\Delta_2 [{}^x_y] = [{}^0_0]$ is equivalent to

$$(T_1 - 3/4) x = T_1 y/2, \qquad (T_1 + 3/4) y = T_1 x/2$$

which implies that $x \in D(T_1^2)$ with

$$(T_1^2 - 9/16) x = T_1^2 x/4, \qquad \text{i.e.,} \quad \tau = \pm\sqrt{3}/2.$$

Accordingly we assume (by virtue of Lemma 2.1 et seq.) that

$$\tau \neq \pm 1, \qquad \tau \neq \pm\sqrt{3}/2. \tag{4.6}$$

Now (4.3) shows that

$$\sigma_p(\Delta_1) = \{\tau - 1, \tau + 1\}, \qquad \sigma_e(\Delta_1) = [\varepsilon - 1, \infty)$$

and so if we assume $\varepsilon > 1$ then

$$\inf \sigma_e(\Delta_1) > 0. \tag{4.7}$$

(Note that we could relate (4.7) to $\varepsilon > \sqrt{3}/2$ which would give $\inf \sigma_e(\Delta_2) > 0$ but this complicates the algebra).

From (4.6), (4.7) we can write

$$\Delta_1 = \Pi_1 - \Phi_1,$$

where Π_1 has a positive bounded inverse and $\Phi_1 \geqslant 0$ with

$$\begin{aligned} \dim \operatorname{rank} \Phi_1 &= 0 \qquad \text{if} \quad \tau > 1 \\ &= 1 \qquad \text{if} \quad -1 < \tau < 1 \\ &= 2 \qquad \text{if} \quad \tau < -1. \end{aligned}$$

We denote by H_0 the Pontryagin Space $D(\Pi_1^{1/2})$ with inner product

$$(x, y)_0 = (\Pi_1^{1/2} x, \Pi_1^{1/2} y) - (x, \Phi_1, y).$$

Then we can define Γ_n as the closure of $\Delta_1^{-1} \Delta_n$ in H_0, $0 \leqslant n \leqslant 2$, so

$$\Gamma_0 = \begin{bmatrix} -(T_1 - I_1)^{-1}/4 & (T_1 - I_1)^{-1}/2 \\ (T_1 + I_1)^{-1}/2 & -(T_1 + I_1)^{-1}/4 \end{bmatrix}\Bigg|_{H_0}.$$

To compute $\sigma_p(\Gamma_0)$ we set

$$\Gamma_0 \begin{bmatrix} y \\ z \end{bmatrix} = \lambda \begin{bmatrix} y \\ z \end{bmatrix}$$

leading to the equations

$$-\frac{y}{4} + \frac{z}{2} = \lambda (T_1 - I_1)\, y,$$

$$\frac{y}{2} - \frac{z}{4} = \lambda (T_1 + I_1)\, z,$$

whence

$$\begin{aligned} (2\lambda T_1 + 1/2 - 2\lambda)\, y &= z \\ (2\lambda T_1 + 1/2 - 2\lambda)\, z &= y. \end{aligned} \tag{4.8}$$

Since Δ_0 is 1–1 we may assume $\lambda \neq 0$ and so $y, z \in D(T_1^2)$ and

$$(2\lambda T_1 + 1/2 + \sqrt{4\lambda^2+1})(2\lambda T_1 + 1/2 - \sqrt{4\lambda^2+1})\, y = 0.$$

It follows that $T_1 y = \tau y$ and so

$$2\lambda\tau + 1/2 \pm \sqrt{4\lambda^2+1} = 0. \tag{4.9}$$

Consequently (4.8) gives $z = (2\lambda(\tau-1)+1/2)\, y$ and so $T_1 z = \tau z$.

We now check the nondegeneracy condition (Assumption 3.2). For this we need to show that

$$0 \neq \delta = \left(\begin{bmatrix} y \\ z \end{bmatrix}, \begin{bmatrix} y \\ z \end{bmatrix} \right)_0 \qquad \text{if} \quad \lambda \in \mathbb{R}.$$

If $\lambda \in \mathbb{R}$ and $\sigma = 0$ then since $y, z \in D(T_1)$ we have

$$\begin{aligned} 0 &= (y, (T_1 - I_1)\, y) + (z, (T_1 + I_1) z) \\ &= [\tau - 1 + (\tau+1)(2\lambda(\tau-1)+1/2)^2](y, y). \end{aligned} \tag{4.10}$$

Thus

$$\begin{aligned} 0 &= \tau - 1 + (\tau+1)(4\lambda^2 + 1 - 4\lambda(2\lambda + 1/2) + 4\lambda^2) \\ &= 2(\tau - 3/4 + \lambda(\tau - 1)) \end{aligned}$$

by repeated use of (4.9).

From (4.10) we find that the above result gives

$$0 = (\tau - 1) + (\tau+1)(2-2\tau)^2 = (\tau-1)(4\tau^2-3),$$
$$\text{i.e.,} \quad \tau = 1 \qquad \text{or} \qquad \tau = \pm\sqrt{3}/2$$

which contradicts (4.6).

With the assumptions of Theorem 3.3 being satisfied, we have the following conclusions.

(i) H_0, which is dense in H, admits a Δ_1-orthogonal decomposition $F \oplus G$, where F, G are invariant under the bounded self-adjoint commuting operators A_n, $0 \leqslant n \leqslant 2$.

(ii) F is finite dimensional and is thus spanned by joint root subspaces of the A_n. In this example (with simple eigenvalues) one can say more. Namely F is spanned by "eigenvectors" $x_1 \otimes x_2, x_m$ $(m = 1, 2)$ satisfying (4.1), (4.2).

(iii) G is Hilbert space (under the H_0 inner product) and thus admits a joint spectral resolution of the form

$$A_n|_G = \int_{\mathbb{R}^3} \mu_n \, dE(\mu). \tag{4.11}$$

The corresponding joint spectral measure has support σ_c in the set σ; see Theorem 3.4 and Fig. 1. To be precise

$$\sigma_c = \sigma \backslash \text{those eigenvalues corresponding to } F.$$

The correspondence is given by

$$(\lambda_1, \lambda_2) = \mu_1^{-1}(1, \mu_2), \qquad (\mu_1, \mu_2, \mu_3) = (\lambda_1^{-1}, 1, \lambda_1^{-1}\lambda_2). \tag{4.12}$$

Since in our example σ_c is "one dimensional", $dE(\mu)$ can actually be parametrized by a single variable. To see this write

$$T_1 = \int_{\mathbb{R}} \lambda \, dS(\lambda)$$

so that

$$A_n|_G = \int_{[\varepsilon, \infty)} M_n(\lambda) \, dS(\lambda),$$

where

$$M_0(\lambda) = \begin{bmatrix} -(\lambda-1)^{-1}/4 & (\lambda-1)^{-1}/2 \\ (\lambda+1)^{-1}/2 & -(\lambda+1)^{-1}/4 \end{bmatrix}, \qquad \text{etc.}$$

The spectral decompositions

$$M_n(\lambda) = \mu_{n-}(\lambda) \, \Pi_-(\lambda) + \mu_{n+}(\lambda) \, \Pi_+(\lambda)$$

provide the formula (4.11); note that the matrices $M_n(\lambda)$ commute for $0 \leqslant n \leqslant 2$. The calculations are simpler if we use matrices corresponding to (4.12), viz.

$$M_0(\lambda)^{-1} M_1(\lambda) = \frac{4}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda+1 \end{bmatrix} = \frac{4}{3}\begin{bmatrix} \lambda-1 & 2(\lambda+1) \\ 2(\lambda-1) & \lambda+1 \end{bmatrix}$$

$$M_0(\lambda)^{-1} M_2(\lambda) = \frac{4}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \lambda-3/4 & -1/2 \\ -\lambda/2 & \lambda+3/4 \end{bmatrix} = \begin{bmatrix} -1 & 2(\lambda+1) \\ 2(\lambda-1) & 1 \end{bmatrix}.$$

The eigenprojectors are still $\Pi_{\pm}(\lambda)$ but the eigenvalues are now precisely the values of λ_1 and λ_2 for $(\lambda_1, \lambda_2) \in \sigma$, viz.

$$\lambda_1^{\pm} = \frac{4}{3}(\lambda \pm \sqrt{4\lambda^2 - 3}), \qquad \lambda_2^{\pm} = \pm\sqrt{4\lambda^2 - 3}.$$

The two signs yield respectively the coordinates of points on the two arcs $P_\varepsilon^- P_\infty^-$ and $P_\varepsilon^+ P_\infty^+$ shown in Fig. 1. For example $\lambda = \varepsilon$ corresponds to $P_\varepsilon^{\pm}$ and $\lambda \to \infty$ corresponds to $\lambda_2^-/\lambda_1^- \to 3/2$, $\lambda_2^+/\lambda_1^+ \to 1/2$.

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